ON A FAILURE TO EXTEND YUDOVICH'S UNIQUENESS THEOREM FOR 2D EULER EQUATIONS

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Abstract. In 1995, Yudovich extended his own 1963 uniqueness result for solutions to the 2D Euler equations with bounded initial vorticity to allow a certain class of initial vorticities whose L^p-norms grow no faster than roughly $\log p$. Yudovich's argument involves estimating part of the difference between two velocities in terms of the L^∞-norm of each velocity. Because the two velocities have a (common) modulus of continuity, however, the L^∞-norm of the difference can be bounded by a function of its L^2-norm, which allows an improvement of this estimate. We show that, though this does, indeed, improve the bound on the difference at time t of the L^2-norm of two solutions having different initial vorticities, it nonetheless does not result in a larger uniqueness class for solutions to the 2D Euler equations.

Contents

We start by describing Yudovich's class of initial velocities in [9] that admit unique solutions to the Euler equations (this extended his bounded vorticity result of [8]). We present Yudovich's uniqueness argument, which is quite short, and describe an obvious improvement. We then explain why this improvement nonetheless does not lead to a larger class of initial velocities.

Let us be clear at the outset, lest there be any confusion, that this is a negative result, which explains the somewhat counterintuitive fact that a simple and obvious extension of a uniqueness argument of Yudovich (Theorem 2.2) achieves, in the end, nothing. It does reflect a small bit of light on the structure of Yudovich's uniqueness class and concave Osgood moduli of continuity, but it is fundamentally a negative result.

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1. Yudovich's uniqueness argument for unbouned initial vorticity

Let v be the velocity field and p the pressure field for an incompressible 2D fluid. We can write the Euler equations in strong form as

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p &= f \quad \text{on } [0,T] \times \Omega, \\ \operatorname{div} v &= 0 \quad \text{on } [0,T] \times \Omega, \\ v &= v^0 \quad \text{on } \{0\} \times \Omega. \end{cases}$$

Here, f is the external force, v^0 the initial velocity, $T > 0$, and $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, the domain in which the fluid lies. When Ω is not all of $\mathbb{R}^d$, we impose the *no-penetration* boundary conditions, $v \cdot \mathbf{n} = 0$ on $\partial\Omega$, where $\mathbf{n}$ is the outward unit normal to the boundary. The external force will play no significant role in our argument, so we will assume it is zero.

The boundary also plays no significant role, and our arguments will work whether $\Omega = \mathbb{R}^2$ or Ω is a bounded domain with C^2-boundary. To simplify the presentation we from now on assume that $\Omega = \mathbb{R}^2$.

Let σ be a *stationary vector field*, meaning that σ is of the form

$$\sigma = \left(-\frac{x_2}{r^2} \int_0^r \rho g(\rho)\, d\rho,\ \frac{x_1}{r^2} \int_0^r \rho g(\rho)\, d\rho \right) \tag{1.1}$$

for some g in $C_C^\infty(\mathbb{R})$ with $\int_{\mathbb{R}^2} g = 1$. For any real number m, a vector v belongs to E_m if it is divergence-free and can be written in the form $v = m\sigma + v'$, where v' is in $L^2(\mathbb{R}^2)$. E_m is an affine space; having fixed the origin, $m\sigma$, in E_m, we can define a norm by $\|m\sigma + v'\|_{E_m} = \|v'\|_{L^2(\Omega)}$. Convergence in E_m is equivalent to convergence in the L^2–norm to a vector in E_m.

For any ω in $L^p(\mathbb{R}^2)$ compactly supported (again for simplicity), $p > 2$, define $K[\omega] = K * \omega$, where

$$K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}. \tag{1.2}$$

(We define $x^\perp = (-x_2, x_1)$.) Then $K[\omega]$ has vorticity ω and lies in the space, E_m, where

$$m = \int_{\mathbb{R}^2} \omega.$$

We will use the definition of a weak solution to the Euler equations in Definition 1.1.

Definition 1.1 (Weak Euler Solutions)**.** Let v^0 be an an initial velocity in E_m. We say that v in $L^\infty(0,T; E_m)$ is a weak solution to the Euler equations (without forcing) if $v(0) = v^0$ and

$$(\mathbf{E}) \qquad \frac{d}{dt} \int_\Omega v \cdot \varphi + \int_\Omega (v \cdot \nabla v) \cdot \varphi = 0$$

for all divergence-free φ in $(H^1(\mathbb{R}^2))^2$.

Suppose that ω is a scalar field lying in L^p for all p in $[1, \infty)$. Let

$$\theta(p) = \|\omega\|_{L^p}, \quad \alpha(\epsilon) = \epsilon^{-1}\theta(\epsilon^{-1}), \tag{1.3}$$

and define

$$\beta(x) = \inf\left\{x^{1-\epsilon}\alpha(\epsilon) \colon \epsilon \text{ in } (0, 1/2]\right\}. \tag{1.4}$$

A classical result of measure theory is that θ is smooth and $p \log \theta(p)$ is convex, so we add this as a requirement, losing no generality in the process.

As shown in Section 10 of [5], β is strictly increasing, concave, twice continuously differentiable, and $\beta(0) = 0$. In particular, this means that β is invertible.

Definition 1.2. We say that ω is a *Yudovich vorticity* if it is compactly supported (this is not essential, but will simplify our presentation) that satisfies the Osgood condition,

$$\int_0^1 \frac{dx}{\beta(x)} = \infty. \tag{1.5}$$

Examples of Yudovich vorticities are

$$\theta_0(p) = 1, \theta_1(p) = \log p, \ldots, \theta_m(p) = \log p \cdot \log^2 p \cdots \log^m p, \tag{1.6}$$

where $\log^m$ is log composed with itself m times. These examples are described in [9] (see also [2].) Roughly speaking, the L^p–norm of a Yudovich vorticity can grow in p only slightly faster than $\log p$. Such growth in the L^p–norms arises, for example, from a point singularity of the type $\log\log(1/|x|)$.

We define the class, $\mathbb{Y}$, of Yudovich velocities to be

$$\mathbb{Y} = \{K[\omega] \colon \omega \text{ is a Yudovich vorticity}\}.$$

A Yudovich velocity will always lie in a space, E_m. Given θ as above, we define the function space,

$$\mathbb{Y}_\theta = \{v \in E_m : \|\omega(v)\|_{L^p} \le C\theta(p) \text{ for all } p \text{ in } [1, \infty)\} \tag{1.7}$$

for some constant C. We define the norm on $\mathbb{Y}_\theta$ to be

$$\|v\|_{\mathbb{Y}_\theta} = \|v\|_{E_m} + \sup_{p \in [p_0, \infty)} \|\omega(v)\|_{L^p} / \theta(p). \tag{1.8}$$

Definition 1.3 (Yudovich Solution)**.** Let v^0 be an an initial velocity in $\mathbb{Y}$. Then a weak solution, v, to the Euler equations is a Yudovich solution if, in addition to the properties in Definition 1.1, $\|\omega(v)(t)\|_{L^p} \le \left\|\omega(v^0)\right\|_{L^p}$ for all p in $[1, \infty)$ and all t in $\mathbb{R}$.

Definition 1.4. We say that a continuous function $f \colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$ is a modulus of continuity (MOC). When we say that a MOC, f, is C^k, $k \ge 0$, we mean that it is continuous on $[0, \infty)$ and C^k on $(0, \infty)$.

A real-valued function or vector field, v, on a normed linear space, X, admits f as a MOC if $|v(x) - v(y)| \le f(\|x - y\|_X)$ for all x, y in X.

Proposition 1.5. *Let v be any Yudovich solution to the Euler equations as in Definition 1.3 with v^0 in $\mathbb{Y}_\theta$. Then $v(t)$ lies in $\mathbb{Y}_\theta$ with $\|v(t)\|_{\mathbb{Y}_\theta} \le \|v^0\|_{\mathbb{Y}_\theta}$ for all t in $\mathbb{R}$. Moreover, $v(t)$ has an Osgood MOC independent of t in $\mathbb{R}$ given (ignoring immaterial constants) by*

$$\mu(x) = \inf\left\{x^{1-2\epsilon}\alpha(\epsilon) \colon \epsilon \text{ in } (0, 1/2]\right\} = \frac{1}{x}\beta(x^2). \tag{1.9}$$

Proof. The uniform-in-time bound on $\|v(t)\|_{\mathbb{Y}_\theta}$ follows immediately from the conservation over time of the L^p-norms of the vorticity. The MOC, μ, follows from potential theory estimates applied to the L^p-norms of the vorticity; that it is Osgood follows by a simple change of variables applied $\beta(x^2)/x$. The form of μ in (1.9) is an in [9], though expressed in a slightly different manner. (Details of the potential theory estimates appear in Section 5.2 of [3]. There is also a much simpler approach using Littlewood-Paley theory, exploiting Bernstein's inequality.) ☐

Theorem 1.6. *Assume that ω^0 lies in $\mathbb{Y}$ with ω^0 compactly supported. Then there exists a Yudovich solution to the Euler equations as in Definition 1.3 with the property that the vorticity is transported by the flow map and*

$$\|\omega(t)\|_{L^p} = \|\omega^0\|_{L^p} \text{ for all } p \text{ in } [1, \infty). \tag{1.10}$$

The value of m in Definition 1.1 is $\int_{\mathbb{R}^2} \omega^0$.

Proof. Existence of a solution, v, for any given fixed p in $(1, \infty]$ is classical (see, for instance, Theorem 4.1 of [6].) That a solution exists for which vorticity is transported by the flow can be proven by modifying the proof of the same result for bounded vorticity in Section 8.2 of [7]. The only significant change is the replacement of the log-Lipschitz MOC that holds for bounded vorticity with the Osgood MOC given by (1.9), and hence the use of Osgood's lemma, Lemma 1.8, in place of Gronwall's inequality. ☐

Theorem 1.7. *If v^0 is in $\mathbb{Y}$ then Yudovich solutions to the Euler equations as in Definition 1.3, whose existence is assured by Theorem 1.6, are unique.*

Proof. Suppose that v_1 and v_2 are two Yudovich solutions to the Euler equations with the same initial velocity, v^0, in $\mathbb{Y}_\theta$. Then a basic energy argument gives

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 \le \int_{\mathbb{R}^2} |\nabla v_2|\, |v|^2 .$$

Letting $M = \|v\|_{L^\infty([0,T]\times\mathbb{R}^2)}$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 &\le \int_{\mathbb{R}^2} |\nabla v_2|\, |v|^{2-2\epsilon} M^\epsilon \le M^\epsilon \|\nabla v_2\|_{L^{\frac{1}{\epsilon}}} \left\| |v|^{2-2\epsilon} \right\|_{L^{\frac{1}{1-\epsilon}}} \\ &= M^\epsilon \|\nabla v_2\|_{L^{\frac{1}{\epsilon}}} \|v\|^{2(1-\epsilon)} \le M^\epsilon \frac{\theta(\epsilon)}{\epsilon} \|v\|^{2(1-\epsilon)}.\end{aligned}$$

Letting $L(t) = \|v(t)\|^2$ and taking the infimum over ϵ in $[1/2, \infty)$ and integrating in time gives

$$L(t) \leq 2 \int_0^t \beta_M(L(s))\, ds,$$

where

$$\begin{aligned}\beta_M(x) &= \inf\left\{M^\epsilon x^{1-\epsilon}\alpha(\epsilon) \colon \epsilon \text{ in } (0, 1/2]\right\} \\ &= M \inf\left\{\left(\frac{x}{M}\right)^{1-\epsilon}\alpha(\epsilon) \colon \epsilon \text{ in } (0, 1/2]\right\} = M\beta\left(\frac{x}{M}\right).\end{aligned}$$

Since β is Osgood, so too is β_M. It follows from Lemma 1.8 that $L \equiv 0$ so that uniqueness holds. □

Using the approximation described in [9], the μ functions corresponding to Yudovich's example vorticities in (1.6) are

$$\mu_k = Cx \log(1/x) x \log^2(1/x) \cdot \log^{k+1}(1/x),\ k = 0, 1, \dots,$$

where $\log^m$ is the logarithm iterated m times. Each corresponding β function, β_k, lies in the same germ at the origin as μ_k, $k = 0, 1, \dots,$ with the dominant term being $2\mu_k$. There is no reason, however, to expect for β and μ to lie in the same germ for an arbitrary Yudovich vorticity.

We used the following version of Osgood's lemma above (see, for instance, p. 92 of [1]).

Lemma 1.8 (Osgood's lemma)**.** *Let L be a measurable nonnegative function and f a nonnegative locally integrable function, each defined on the domain $[t_0, t_1]$. Let $\mu \colon [0, \infty) \to [0, \infty)$ be a continuous nondecreasing function, with $\mu(0) = 0$. Let $a \geq 0$, and assume that for all t in $[t_0, t_1]$,*

$$L(t) \leq a + \int_{t_0}^t f(s)\mu(L(s))\, ds. \tag{1.11}$$

If $a > 0$, then

$$\int_a^{L(t)} \frac{ds}{\mu(s)} \leq \int_{t_0}^t f(s)\, ds.$$

If $a = 0$ and $\int_0^\infty ds/\mu(s) = \infty$, then $L \equiv 0$.

2. Extending Yudovich's uniqueness class

In the proof of Theorem 1.7, above, we simply pulled out the L^∞-norm of u and used the fact that it is bounded uniformly over finite time. In fact, though, because u is the difference between two vector fields, u_1 and u_2, each having the same MOC, its L^∞-norm can be bounded in terms of its L^2-norm. The idea, which is formalized in Lemma 2.1 (taken from Lemma 8.3 of [4]), is that if u_1 and u_2 differ at a point, they must differ significantly in a ball around that point, for the rate at which the difference can drop to zero as one moves away from the point is limited by the MOC of u_1 and u_2.

Lemma 2.1. *Let u_1, u_2 be two velocity fields on $\mathbb{R}^2$ each having MOC μ. Then*

$$\|u_1 - u_2\|_{L^\infty} \le F\left(\|u_1 - u_2\|\right),$$

where (ignoring immaterial constants),

$$F(\cdot) = (\cdot\mu^{-1}(\cdot))^{-1}.$$

Observe that $F(0) = 0$ and that for u_1, u_2 Yudovich velocities—which are in L^∞—F is bounded.

To express F in a more convenient manner, let $h(\cdot) = \cdot\mu^{-1}(\cdot)$ and $\delta = F(x) = h^{-1}(x)$. It follows that $x = h(F(x)) = F(x)\mu^{-1}(F(x)) = \delta\mu^{-}(\delta)$ so that

$$\mu\left(\frac{x}{\delta}\right) = \delta.$$

But using (1.9) this becomes

$$\mu\left(\frac{x}{\delta}\right) = \frac{\delta}{x}\beta\left(\frac{x^2}{\delta^2}\right) = \delta \implies \beta\left(\frac{x^2}{F(x)^2}\right) = \beta\left(\frac{x^2}{\delta^2}\right) = x$$

so that

$$F(x) = \frac{x}{\sqrt{\beta^{-1}(x)}}. \tag{2.1}$$

To extend the space, $\mathbb{Y}$, we start, as in the definition of $\mathbb{Y}$, with θ, α, and β as defined in (1.3, 1.4) along with μ as defined in (1.9), and define F as in (2.1). We then define

$$\gamma(x) = \inf\left\{x^{1-\epsilon}F(\sqrt{x})^{2\epsilon}\alpha(\epsilon) \colon \epsilon \text{ in } (0, 1/2]\right\}$$

and define the extended Yudovich space, $\mathbb{Y}'$, to be all velocity fields corresponding to compactly supported vorticities, ω, for which γ is Osgood. That is, if ω has a corresponding γ that is Osgood, then $K[\omega]$ lies in $\mathbb{Y}'$. We then define $\mathbb{Y}'_\theta$ in analogy with (1.7).

Because F is bounded, it is clear that $\mathbb{Y} \subseteq \mathbb{Y}'$. But in fact, the two spaces are identical, which is not immediately obvious. We show this in Section 3.

The obvious modification to the proof of Theorem 1.7 is quite simple, and leads to Theorem 2.2.

Theorem 2.2. *If v^0 is in $\mathbb{Y}'$ then weak solutions having the properties in Definition 1.3 are unique.*

Proof. We argue as in the proof of Theorem 1.7, though we use Lemma 2.1 to bound $\|u\|_{L^\infty}$. This leads to

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 \le \|\nabla v_2\|_{L^{\frac{1}{\epsilon}}}\|v\|^{2(1-\epsilon)}M^\epsilon \le \frac{\theta(\epsilon)}{\epsilon}\|v\|^{2(1-\epsilon)}F(\|v\|)^{2\epsilon},$$

where $M = \|v\|_{L^\infty([0,T]\times\mathbb{R}^2)}$. Letting $L(t) = \|v(t)\|^2$ and taking the infimum over ϵ in $[1/2, \infty)$ and integrating in time gives

$$L(t) \le 2\int_0^t \gamma(L(s))\,ds.$$

Since γ is Osgood, it follows from Lemma 1.8 that $L \equiv 0$ so that uniqueness holds. $\square$

3. The extension is no extension

We can simplify the expression for γ considerably. We have,

$$\begin{aligned}\gamma(x) &= F(\sqrt{x})^2 \inf\left\{\left(\frac{x}{F(\sqrt{x})^2}\right)^{1-\epsilon}\alpha(\epsilon)\colon \epsilon \text{ in } (0,1/2]\right\}\\ &= F(\sqrt{x})^2\beta\left(\frac{x}{(F(\sqrt{x})^2}\right).\end{aligned}$$

But from (2.1),

$$F(\sqrt{x})^2 = \left(\frac{\sqrt{x}}{\sqrt{\beta^{-1}(\sqrt{x})}}\right)^2 = \frac{x}{\beta^{-1}(\sqrt{x})}$$

so that

$$\gamma(x) = \frac{x}{\beta^{-1}(\sqrt{x})}\beta(\beta^{-1}(\sqrt{x})) = \frac{x^{3/2}}{\beta^{-1}(\sqrt{x})}. \tag{3.1}$$

Observe then that, by (3.1),

$$\int_0^1 \frac{dx}{\gamma(x)} = \int_0^1 \frac{\beta^{-1}(\sqrt{x})}{x^{3/2}}\,dx.$$

Making the change of variables, $z = \beta^{-1}(\sqrt{x})$ so that $x = \beta(z)^2$, $dx = 2\beta(z)\beta'(z)\,dz$, and noting that $\beta^{-1}(0) = 0$, we have

$$\int_0^1 \frac{dx}{\gamma(x)} = 2\int_0^{\beta^{-1}(1)} \frac{z\beta(z)\beta'(z)}{\beta(z)^3}\,dz = 2\int_0^{\beta^{-1}(1)} \frac{z\beta'(z)}{\beta(z)^2}\,dz.$$

Thus, we have proved Proposition 3.1.

Proposition 3.1. *The velocity field, u, lies in $\mathbb{Y}'$ if and only if*

$$\int_0^1 \frac{x\beta'(x)}{\beta(x)^2}\,dx = \int_0^1 \frac{x(\log\beta)'(x)}{\beta(x)}\,dx = \infty.$$

Now suppose that that u lies in $\mathbb{Y}' \setminus \mathbb{Y}$. Then it must be that

$$\int_0^1 \frac{dx}{\beta(x)} < \infty \text{ while } \int_0^1 \frac{x(\log\beta)'(x)}{\beta(x)}\,dx = \infty,$$

the first condition excluding membership of u in $\mathbb{Y}$, the second including its membership in $\mathbb{Y}'$ by Proposition 3.1.

But by Proposition 10.2 of [5], $0 < x(\log\mu)'(x) \le 1$ for all $x > 0$, so by (1.9),

$$\begin{aligned} 0 < x\left[\log\left(\frac{1}{x}\beta(x^2)\right)\right]' &= x\left[-\log x + (\log\beta(x^2))'\right] \\ &= x\left[-\frac{1}{x} + 2x(\log\beta)'(x^2)\right] = -1 + 2x^2(\log\beta)'(x^2) \le 1. \end{aligned}$$

It follows that

$$\frac{1}{2} < x(\log\beta)'(x) \le 1 \tag{3.2}$$

so that

$$\int_0^1 \frac{x(\log\beta)'(x)}{\beta(x)}\,dx \le \int_0^1 \frac{dx}{\beta(x)} < \infty. \tag{3.3}$$

Hence, there can be no such velocity field in $\mathbb{Y}' \setminus \mathbb{Y}$, and we conclude that $\mathbb{Y}' = \mathbb{Y}$.

Notice that (3.3) does not contradict $\gamma(x) < \mu(x)$ for all sufficiently small x, as must be the case since $F(0) = 0$. This is because Proposition 3.1 was in terms of β rather than μ and involved a change of variables in the integral as well.

4. An explicit example

We suppose that we have a velocity field for which $\theta(p) = p^a$, $a \ge 1$, noting that $p\log\theta(p)$ is, as we required, convex. Such a velocity field lies far outside of $\mathbb{Y}$ in that the L^p-norms of its vorticity can grow as fast as any polynomial. What is perhaps surprising is that the resulting functions, μ and β, are not much larger (as these things go) than those corresponding to the examples of Yudovich in (1.6), which are all close to $-x\log x$. This illustrates just how delicate selection for membership in $\mathbb{Y}$ can be.

As an example, suppose that $\theta(p) = p^a$, $a \ge 1$, noting that $p\log\theta(p)$ is, as we required, convex. Then $\alpha(\epsilon) = (1/\epsilon)^a/\epsilon = \epsilon^{-a-1}$, so

$$\mu(x) = \inf\left\{x^{1-2\epsilon}\epsilon^{-a-1} \colon \epsilon \text{ in } (0, 1/2]\right\}.$$

Fixing $x > 0$, let $f(x) = x^{1-2\epsilon}\epsilon^{-a-1}$. Then

$$f'(\epsilon) = -x^{1-2\epsilon}\epsilon^{-a-2}\left[2\epsilon\log x + a + 1\right].$$

Noting that $f(0^+) = f(\infty) = \infty$, the infimum of $f(\epsilon)$ occurs when $2\epsilon\log x + a + 1 = 0$; that is, when

$$\epsilon = \epsilon_0 := -\frac{a+1}{2\log x}.$$

Therefore, as long as x is small enough that $\epsilon_0 \le 1/2$,

$$\mu(x) = x^{1-2\epsilon_0}\epsilon_0^{-a-1}.$$

Since

$$x^{-2\epsilon_0} = e^{-2\epsilon_0 \log x} = e^{a+1},$$

we can write

$$\mu(x) = e^{a+1} x \left[-\frac{2 \log x}{a+1} \right]^{a+1},$$

which we note is not Osgood since $a \geq 1$.

Using (1.9), both $\mu(x)$ and $\beta(x)$ are constant multiples of $(-\log x)^{a+1} x$. Ignoring immaterial constants, then,

$$(\log \beta)'(x) = (\log x + (a+1) \log(-\log x))' = \frac{1}{x} + \frac{a+1}{x \log x}$$

so

$$\begin{aligned} \int_0^1 \frac{x(\log \beta)'(x)}{\beta(x)} \, dx &= \int_0^1 \frac{1 + (a+1)/\log x}{x(-\log x)^{a+1}} \, dx \\ &= \int_0^1 \frac{1}{x(-\log x)^{a+1}} \, dx - (a+1) \int_0^1 \frac{1}{x(-\log x)^{a+2}} \, dx < \infty. \end{aligned}$$

Therefore, since also $\int_0^1 (\beta(x))^{-1} \, dx < \infty$, the velocity field lies neither in $\mathbb{Y}$ nor in $\mathbb{Y}'$.

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